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Non-topological solitons in a U(1) gauge theory

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Abstract. We construct a new class of non-topological solitons (NTSs) in a renormalizable scalar field theory with a U(1) gauge symmetry. In the thin-walled limit, we show an explicit solution of gauged NTS. The soliton consists of an interior region of false vacuum supported against collapse by the pressure of massless particles trapped in its interior. For a range of values of the gauge coupling e the soliton is stable and becomes a superconductor. For large charge Q, there exist $Q_{\max}$ and $Q_{\min}$ in the $e<e_{\text{crit}}$ case.

1. Introduction

Non-topological solitons (NTSs) are solutions of classical field theories which are stable by virtue of a conserved Noether charge carried by fields confined to a finite region of space [1]. In the past few years, these solutions have been studied under the guise of Q-balls [2], cosmic neutrino balls [3], quark nuggets [4] and soliton stars [5], and a scenario for producing them in a phase transition in the early universe has been considered [6]. The simplest example of an NTS is the Q-ball that can appear in a U(1)-invariant theory with a complex scalar field that has nonlinear self-interactions. Recently, the gauged Q-balls have been studied in a local U(1) theory by Lee *et al* [7]. This work provides a possibility for understanding how NTSs might arise in realistic gauge theories such as electromagnetism, or unified theories. These workers have considered the following form for the potential:

$$U(f)=\frac{\lambda^2 f^6}{6\mu^2}-\frac{f^4}{4}+\frac{\mu^2 f^2}{2} \tag{1}$$

where λ is a dimensionless constant and μ is the mass of a free ϕ particle. This potential is non-renormalizable. One must consider sixth (or higher) order potentials, since a necessary condition for the existence of Q-balls is that the function $U(f)/f^2$ has a minimum for some non-zero f. The simplest renormalizable theory with NTS solutions is an unbroken U(1) theory of two coupled scalar fields: ϕ is complex and h is Hermitian [8]. In this paper, we study the classical NTS solutions of this theory in a local U(1) theory. It may provide a better understanding of the connection between gauge theories and NTSs.

Consider the Lagrangian for a Hermitian scalar field $h(\boldsymbol{r}, t)$ and a complex scalar field $\phi(\boldsymbol{r}, t)=f(\boldsymbol{r}, t)\exp(\mathrm{i}\theta(\boldsymbol{r}, t))/\sqrt{2}$ which is coupled to a U(1) gauge field A_μ:

$$\mathscr{L}=\tfrac{1}{2}(\partial_\mu f)^2+\tfrac{1}{2}f^2(\partial_\mu\theta-eA_\mu)^2+\tfrac{1}{2}(\partial_\mu h)^2-U(\phi, h)-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2}$$

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where $D_\mu = \partial_\mu - ieA_\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. (For definiteness, we take $e > 0$.) The interaction invariant under the discrete symmetry $h \to -h$ and the U(1) symmetry $\phi \to e^{i\theta}\phi$ is

$$U(\phi, h) = \alpha^2 h^2 |\phi|^2 + \tfrac{1}{8}\beta^2 (h^2 - h_0^2)^2. \tag{3}$$

The conserved charge associated with the U(1) symmetry is

$$Q = \int d^3 r\, f^2 \left(\frac{\partial \theta}{\partial t} - eA_0 \right). \tag{4}$$

Spherically symmetric NTSs solutions for the theory of (2) and (3) were first studied by Friedberg *et al* [8] for the special case $e = 0$. For large Q, it is characterized by an interior false vacuum region with $h = 0$, surrounded by a thin domain wall where h rapidly approaches its ground state value $h = h_0$. In the NTS interior, the potential energy density in h is balanced by the pressure of the massless ϕ charges, which are confined by the mass gap αh_0 at the domain wall. As a simple consequence of the symmetry breaking for the $e \neq 0$ case, the gauge field is massive inside and NTS is a U(1) superconductor.

2. Qualitative properties of soliton

We begin by deriving the gauged NTSs for the $e \neq 0$ case as a natural extension of the derivation in [7]. We expect gauged NTSs to be stable as long as their electrostatic self-energy is much smaller than the other energies. Consider a coherent configuration of ϕ, h and A_μ with a given electric charge eQ. The lowest energy solutions are of the spherically symmetric form [1, 7]:

$$\begin{aligned} &\phi(\boldsymbol{r}, t) = e^{i\omega t} f(r)/\sqrt{2} && h(\boldsymbol{r}, t) = h(r) \\ &A_\mu(r) = 0 \qquad (\mu \neq 0) && A_0 = (\omega - g(r))/e \end{aligned} \tag{5}$$

where we assume $\omega > 0$ for definiteness. The lowest-energy state will have no electric currents and therefore no magnetic fields. The spatial components of the gauge potential are zero as there is no magnetic field. We choose a boundary condition $A_0(r) \to 0$ as $r \to \infty$.

The Lagrangian for the configuration described above is

$$L = 4\pi \int r^2\, dr \left(-\tfrac{1}{2} f'^2 + \frac{1}{2e^2} g'^2 - \tfrac{1}{2} h'^2 - U(f, h) + \tfrac{1}{2} f^2 g^2 \right) \tag{6}$$

where the prime denotes d/dr. By varying L with respect to f, h and g at fixed ω we find the equations of motion:

$$f'' + \frac{2}{r} f' + f g^2 - \frac{\partial U(f, h)}{\partial f} = 0 \tag{7}$$

$$g'' + \frac{2}{r} g' - e^2 f^2 g = 0 \tag{8}$$

$$h'' + \frac{2}{r} h' - \frac{\partial U(f, h)}{\partial h} = 0. \tag{9}$$

The energy functional for the solution (5) can be written as

$$E = 4\pi \int r^2\, dr \left(\tfrac{1}{2} f'^2 + \frac{1}{2e^2} g'^2 + \tfrac{1}{2} h'^2 + \tfrac{1}{2} f^2 g^2 + U(f, h) \right). \tag{10}$$

The soliton charge is

$$Q=4\pi\int r^2\,\mathrm{d}r\,f^2 g. \tag{11}$$

To gain insight into the gauged NTS solitons, we show some qualitative properties of the soliton solution. From (8) and (11) we see that

$$e^2Q=\lim_{r\to\infty}4\pi r^2 g'. \tag{12}$$

For large r, $g\to\omega-e^2Q/4\pi r$, f is small, h approaches the constant h_0, and $U(f,h)\simeq m^2f^2/2$ $(m=\alpha^2h_0^2)$. Equation (7) is reduced to

$$f''+\frac{2}{r}f'+(g^2-m^2)f=0 \tag{13}$$

and $f\propto\exp(-\sqrt{m^2-\omega^2}\,r)/r$. Clearly, a necessary condition for the existence of a solution is $\omega<m$. Additionally, for the soliton to be well behaved at the origin, f', h' and g' should approach zero at least faster than r for $r\to0$. By using the asymptotic behaviour of f, h and g, the energy functional can be written as

$$E=\tfrac{1}{2}\omega Q+4\pi\int r^2\,\mathrm{d}r\left(\tfrac{1}{2}f'^2+\tfrac{1}{2}h'^2+U(f,h)\right). \tag{14}$$

Non-topological solitons are quantum mechanically stable if they are the lowest-energy configuration of fixed charge. For the $e=0$ case, we know that $E\simeq(4\pi/3)m(2\beta^2)^{-1/4}Q^{3/4}$ for large Q so that for $Q>Q_s$ the lowest-energy state is favoured over the free particle one ($E_{\text{free}}=\mu Q$). For the $e\neq0$ case, we expect that the energy will be increased over the $e=0$ case due to Coulomb repulsion with Coulomb energy becoming more important as Q becomes large. However, for the $\partial E/\partial Q>\mu$ case, we must consider that some of the charge can be put into the interior region of the soliton and some can be put in free particles. As discussed in [7], there exists a maximum charge $Q_{\max}$ such that $Q>Q_{\max}$, an NTS with charge $Q_{\max}$ plus $Q-Q_{\max}$ free particles will be the lowest-energy state for the system.

3. Thin-walled soliton

Spherically symmetric trial functions are used:

$$f(r)=\begin{cases}F(r) & r\leqslant R\\ 0 & r\geqslant R\end{cases} \tag{15}$$

$$g(r)=\begin{cases}G(r) & r\leqslant R\\ \omega-(e^2Q)/(4\pi r) & r\geqslant R\end{cases} \tag{16}$$

$$h(r)=\begin{cases}0 & r\leqslant R\\ h_0\{1-\exp[-(r-R)/L]\} & r\geqslant R\end{cases} \tag{17}$$

where R and L are two length parameters. R is the radius of the soliton, given by the first zero of $F(r)$. L is the thickness of the domain wall separating the $h=0$ interior region of false vacuum from the $h=h_0$ exterior region of true vacuum. For large Q (width of the shell much less than the radius of the NTS) the energy associated with the shell is negligible. In the thin-wall limit $\to 0$, we can treat h as a step function. The equations of motion (7)-(9) are reduced to

$$F''+\frac{2}{r}F'+FG^2=0 \tag{18}$$

$$G''+\frac{2}{r}G'-e^2F^2G=0. \tag{19}$$

One can find the power-series solutions for F and G:

$$F=a\sum_{k=0}^{\infty}\frac{(-1)^k f_{2k}r^{2k}}{(2k+1)!} \tag{20}$$

$$G=b\sum_{k=0}^{\infty}\frac{g_{2k}r^{2k}}{(2k+1)!} \tag{21}$$

where

$$\begin{gathered}f_0=1 \qquad f_2=b^2 \qquad f_4=b^4-2e^2a^2b^2 \qquad f_6=b^6-\tfrac{38}{3}e^2a^2b^4+\tfrac{16}{3}e^4a^4b^2\dots\\ g_0=1 \qquad g_2=e^2a^2 \qquad g_4=e^4a^4-2e^2a^2b^2\\ g_6=e^6a^6-\tfrac{38}{3}e^4a^4b^2+\tfrac{16}{3}e^2a^2b^4\dots.\end{gathered} \tag{22}$$

Furthermore, we show that the recursion formulae of the coefficients f_{2k} and g_{2k} can be expressed as follows:

$$f_{2m+2}=(2m+1)!\sum_{n=0}^{m}\sum_{k=0}^{n}\frac{(-1)^n g_{2k}}{(2k+1)!}\frac{g_{2(n-k)}}{(2n-2k+1)!}\frac{f_{2(m-n)}}{(2m-2n+1)!}b^2 \tag{23}$$

$$g_{2m+2}=(2m+1)!\sum_{n=0}^{m}\sum_{k=0}^{n}\frac{(-1)^n f_{2k}}{(2k+1)!}\frac{f_{2(n-k)}}{(2n-2k+1)!}\frac{g_{2(m-n)}}{(2m-2n+1)!}e^2a^2. \tag{24}$$

The function $F(r)$ may be written as a sum of two terms, the first part is independent upon e and the second part is dependent upon e:

$$F(r)=\frac{a}{br}\sin br-\frac{2e^2a^3b^2r^4}{5!}+\frac{(\frac{16}{3}e^4a^5b^2-\frac{38}{3}e^2a^3b^4)r^6}{7!}-\dots. \tag{25}$$

Spherically symmetric NTS solutions were first studied by Friedberg *et al* [8] for the special case $e=0$. The solutions (15)-(25) are a more general class of solutions.

In the $e\ll 1$ case, we have the asymptotic solution

$$f=\begin{cases}\pi\sqrt{Q/2}[\sin(\omega-(e^2Q)/(4\pi R))r]/r & r\leqslant R\\ 0 & r\geqslant R\end{cases} \tag{26}$$

$$g=\begin{cases}\omega-(e^2Q)/(4\pi R) & r\leqslant R\\ \omega-(e^2Q)/(4\pi r) & r\geqslant R\end{cases} \tag{27}$$

$$h=\begin{cases}0 & r\leqslant R\\ h_0 & r\geqslant R.\end{cases} \tag{28}$$

Substituting (26)-(28) into (14), we get

$$E=\frac{\pi Q}{R}+\frac{\pi R^3m^4}{6\beta^2}+\frac{Q^2e^2}{8\pi R}. \tag{29}$$

The last term is the electrostatic self-energy of the NTS in the region $r \geq R$. The term $4\pi \int_0^R r^2 \, dr \, G'^2$ is neglected in the asymptotic solution case, which corresponds to the electrostatic self-energy of the NTS in the region $r \leq R$.

For fixed Q, minimizing the energy with respect to R gives

$$R_{\min} \simeq (2Q\beta^2)^{1/4}(1+(Qe^2)/(8\pi^2))^{1/4}/m. \tag{30}$$

At this radius, the energy of the NTS is given by

$$E(R_{\min}) \simeq \frac{4\pi m}{3}(2\beta^2)^{-1/4} Q^{3/4}\left(1+\frac{Qe^2}{8\pi^2}\right)^{3/4}. \tag{31}$$

For ordinary $(e=0)$ NTSs, (31) becomes

$$E \simeq \frac{4\pi m}{3}(2\beta^2)^{-1/4} Q^{3/4}. \tag{32}$$

This is less than the plane-wave solution $(E = Q\mu)$ when $Q > Q_s \sim (4\pi m/3\mu)^4/3\beta^2$. Hence, when $Q > Q_s$ the solution exists and is absolutely stable. For gauging $(e \neq 0)$ NTSs, we consider the condition $E(R_{\min})/Q\mu < 1$ so that the solitons would be stable against dispersion into free particles, i.e.

$$Q^{-1/4}\left(1+\frac{Qe^2}{8\pi^2}\right)^{3/4} < \frac{3}{4\pi}\frac{\mu}{m}(2\beta^2)^{1/4}. \tag{33}$$

The values of Q depend on e, and $(\mu/m)(2\beta^2)^{1/4}$, which are model-dependent parameters.

From (33) we have

$$Q_{\min} = \frac{d^2}{c}\sqrt{\frac{4}{27c}}\cos\left[\tfrac{1}{3}\cos^{-1}\left(-\frac{1}{d^2}\sqrt{\frac{27c}{4}}\right)\right] - \frac{1}{c} \tag{34}$$

$$Q_{\max} = \frac{d^2}{c}\sqrt{\frac{4}{27c}}\cos\left[\tfrac{1}{3}\cos^{-1}\left(-\frac{1}{d^2}\sqrt{\frac{27c}{4}}\right) + \frac{4}{3}\pi\right] - \frac{1}{c} \tag{35}$$

where

$$c = \frac{e^2}{8\pi^2} \tag{36}$$

and

$$d = \frac{3}{4\pi}\frac{\mu}{m}(2\beta^2)^{1/4}. \tag{37}$$

The fact that there is a maximum value for the allowed charge of a gauged NTS points to a fundamental difference with the ordinary $(e=0)$ case. In other words, there is a maximum size for a gauged NTS, making impossible the existence of the solution in bulk form. As mentioned above, stable NTSs must have $Q_{\min} < Q < Q_{\max}$.

There exists a critical value of e above which there is no solution to the equation defining $Q_{\max}$ and $Q_{\min}$. From (34)–(37), it is easy to find that

$$e_{\mathrm{crit}} = \frac{\sqrt{3}\,\beta}{2\pi}\left(\frac{\mu}{m}\right)^2. \tag{38}$$

We see that the NTSs can occur when $e < e_{\mathrm{crit}}$.

We have studied the renormalizable model containing a non-topological soliton in a U(1) gauge theory; many physical features of the solution discussed here are very similar to the models which inspired it.

References

[1] Lee T D 1976 *Phys. Rep.* **23C** 254
[2] Colemann S 1985 *Nucl. Phys.* B **262** 263
[3] Holdom B 1987 *Phys. Rev.* D **36** 1000
[4] Witten E 1984 *Phys. Rev.* D **30** 272
[5] Lee T D 1987 *Phys. Rev.* D **35** 3637
[6] Frieman J, Gelmini G, Gleiser M and Kolb E 1988 *Phys. Rev. Lett.* **60** 2101
[7] Lee K, Stain-Schabes J A, Watkin R and Widrow L M 1989 *Phys. Rev.* D **39** 1665
[8] Friedberg R, Lee T D and Sirlin A 1976 *Phys. Rev.* D **13** 2739